# NOTES ON COMBINATORIAL SET THEORY

BY

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ABSTRACT

We shall prove some unconnected theorems: (1) (G.C.H.) $\omega_{\alpha+1} \to (\omega_\alpha + \xi)^2_2$ *when* $\aleph_\alpha$ *is regular,* $|\xi|^+ < \omega_\alpha$. (2) *There is a Jonsson algebra in* $\aleph_{\alpha+n}$, *and* $\aleph_{\alpha+n} \nrightarrow [\aleph_{\alpha+n}]^{n+1}_{\aleph_{\alpha+n}}$ *if* $2^{\aleph_\alpha} = \aleph_{\alpha+n}$. (3) *If* $\lambda > \aleph_0$ *is a strong limit cardinal, then among the graphs with* $\leqq \lambda$ *vertices each of valence* $< \lambda$ *there is a universal one.* (4) (G.C.H.) *If f is a set mapping on* $\omega_{\alpha+1}$ ($\aleph_\alpha$ *regular*) $|f(x) \cap f(y)| < \aleph_\alpha$, *then there is a free subset of order-type* $\xi$ *for every* $\xi < \omega_{\alpha+1}$.

## 1. Introduction

We shall solve here some infinite combinatorial problems, most of which appeared in Erdös and Hajnal [3, 4]. The definitions appear in the appropriate sections, and the background in the introduction. Two of the results appeared in my notice [15].

Section 2 solves as a particular case a problem of Hajnal [6] which is Problem 36 of [3]. We prove (G.C.H.) that *if f* is a set mapping on $\omega_{\alpha+1}$, $\aleph_\alpha$ is regular, and $x \neq y \in \omega_{\alpha+1} \to |f(x) \cap f(y)| < \aleph_\alpha$, *then* for every $\xi < \omega_{\alpha+1}$, $\omega_{\alpha+1}$ has a free subset of order-type $\xi$. Hajnal's question was for $\alpha = 0$ (and then G.C.H. is not needed) and its solution follows from the strong theorem $\omega_1 \to (\xi)^2_n$ ($n < \omega$, $\xi < \omega_1$) of Baumgartner and Hajnal [1], but this does not imply our general result. In [4] another generalization of Problem 36 of [3], due to Prikry, is mentioned:

THEOREM (Prikry). *Let* $[\omega_1]^2 = I_0 \cup I_1$, $I_0 \cap I_1 = \emptyset$. *If there are no sets*

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*$A, B \subset \omega_1$ such that $|A| = \aleph_0$, $|B| = \aleph_1$ and $A \otimes B \subseteq I_0$, then for every $\alpha < \omega_1$ there is a set $C$ of type $\alpha$ such that $[C]^2 \subset I_1$.*

In §3 we shall prove that for every strong limit singular cardinal $\lambda > \aleph_0$ (that is $\mu < \lambda \to 2^\mu < \lambda$), in $K_\lambda$ there is a universal graph. ($K_\lambda$ is the set of graphs with $\lambda$ vertices such that the valency of each vertex is $< \lambda$. A graph $G$ is universal in $K_\lambda$ if $G \in K_\lambda$, and every graph in $K_\lambda$ is isomorphic to a subgraph of $G$ spanned by its set of vertices.) This solves Problem 74 (from [3]) of Erdös and Rado. Rado in [12, 13] proved (G.C.H.) that for every regular $\aleph_\alpha > \aleph_0$, $K_{\aleph_\alpha}$ has a universal graph. He also proved (G.C.H.) that for regular $\aleph_\alpha \geqq \aleph_0$, $K'_{\aleph_\alpha}$ has a universal graph, where $K'_{\aleph_\alpha}$ is the set of graphs with $\leqq \aleph_\alpha$ vertices. De Bruin proved $K_{\aleph_0}$ does not have a universal graph. Rotman [14] discusses the connection between Rado's theorem, and general model-theoretic theorems of Jonsson [8] and Morley and Vaught [10] on the existence of universal (and homogeneous) models. Together with Rado's theorem, our result implies that (G.C.H.) for every $\alpha > 0$, $K_{\aleph_\alpha}$ has a universal graph.

We ask another natural question: is there a strongly universal graph, in the sense that the embedding of each $G^* \in K_\lambda$ into it, preserves valency? We answer this positively if $\lambda = \aleph_\alpha > |\alpha| + \aleph_0$ and G.C.H. holds. If $\aleph_\alpha = \alpha$, the answer is negative.

It remains an open question whether $\aleph_0 + |\alpha| < \aleph_\alpha$ and G.C.H. are needed. Similar questions remain open for the existence of a universal graph in $K_\lambda$ and $K'_\lambda$.

In §4, we prove that if $2^{\aleph} \leqq \aleph_{\alpha+n}$ then $\aleph_{\alpha+n} \nrightarrow [\aleph_{\alpha+n}]^{n+1}_{\aleph_{\alpha+n}}$ and there is a Jonsson algebra of cardinality $\aleph_{\alpha+n}$. (I first proved this using the stronger assumption that $2^{\aleph} = \aleph_{\alpha+n}$, and Galvin observed that the same proof works if one assumes only that $2^{\aleph_\alpha} \leqq \aleph_{\alpha+n}$.) This generalizes the theorem of Erdös and Hajnal [5] that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ implies $\aleph_{\alpha+1} \nrightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$ and the theorem of Erdös and Hajnal [2] and Chang that then there is a Jonsson algebra in $\aleph_{\alpha+1}$.

In §5 we prove a combinatorial lemma which will be used in §6, and hopefully it will have further application. Its intuitive meaning is that for suitable colorings of $\omega_{\alpha+1}$, there is a subset of the same cardinality in which appear only the colors appearing "heavily" in $\omega_{\alpha+1}$.

In §6 we prove (G.C.H.) that if $\aleph_\alpha$ is regular, $|\xi|^+ < \aleph_\alpha$, then $\omega_{\alpha+1} \to (\omega_\alpha + \xi)^2_2$. This generalizes an unpublished result of Erdös and Hajnal: $\omega_2 \to (\omega_1 + n)^2_2$ (see [4, Problem 10]). It should be noted that Hajnal, by a slight generalization of

Prikry [11] proved, in fact, that ZFC + GCH is consistent with $\omega_2 \nrightarrow (\omega_1 + \omega)^2$ and even $\omega_2 \nrightarrow [\omega_1 + \omega]^2_{\aleph_1}$. It remains an open question whether $\omega_2 \rightarrow (\omega_1 + 2)^2_3$.

NOTATION. The cardinality of a set $A$ is $|A|$ and the order type of $(A, <)$ is denoted by $\mathrm{tp}(A, <)$ or simply by $\mathrm{tp}\,A$ if the order relation is understood. We identify an ordinal with the set of smaller ordinals, the $\alpha$'th infinite cardinal is $\aleph_\alpha$, and the corresponding initial ordinal is $\omega_\alpha$. Ordinals will be $i, j, k, l, \alpha, \beta, \gamma, \delta, \xi, \zeta$ ($\delta$ a limit ordinal), cardinals will be $\lambda, \mu, \kappa$, and natural numbers will be $m, n$. Let $\lambda^{\bar{\mu}} = \sum_{k<\mu} \lambda^k$. Sets will be denoted by $A, B, S, X, Y, Z$ and elements of sets (which are frequently ordinals) are denoted by $a, b, v, x, y$. $[A]^n$ is the set of all subsets of $A$ with exactly $n$ elements. If $A, B, \subseteq S$, $x \in S$, $S$ is ordered by $<$, then $A < B$ means $y \in A$, $z \in B \rightarrow y < z$; $A < x$ means $y \in A \rightarrow y < x$, etc.

## 2. The existence of free subsets

DEFINITION 2.1. A) A function $f$ is a set mapping (of type 1) on a set $S$ if its domain is $S$, and for any $a \in S$, $f(a) \subseteq S$, $a \notin f(a)$.

B) A subset $S_1$ of $S$ is free (for $f$) if for any distinct $a, b \in S_1$, $a \notin f(b)$.

THEOREM 2.1. *If $f$ is a set mapping on $\lambda^+$, $\lambda = \lambda^{\bar{\mu}}$, and for any distinct $x, y \in \lambda^+$ $|f(x) \cap f(y)| < \lambda$, then for any ordinal $\xi < \mu^+$, $\lambda^+$ has a free subset of order type $\xi$.*

PROOF. If there is $x \in \lambda^+$, $|f(x)| = \lambda^+$, then $\langle f(x), < \rangle$ satisfies: $\mathrm{tp}\langle f(x), <\rangle = \lambda^+$, and for $y \in f(x)$, $|f(y) \cap f(x)| < \lambda$, hence by Hajnal [7] there is $S \subseteq f(x)$, $\mathrm{tp}\langle S, < \rangle = \lambda^+$, such that for any $y \in S$, $[f(y) \cap f(x)] \cap S = \emptyset$, hence $f(y) \cap S = f(y) \cap (f(x) \cap S) = [f(y) \cap f(x)] \cap S = \emptyset$. So $S$ is a free subset of $\lambda^+$ of order-type $\lambda^+$, and we get more than we want. Hence w.l.o.g.

$$|f(x)| \leqq \lambda \text{ for any } x \in \lambda^+. \tag{1}$$

Let us define a function $F_1 : \lambda^+ \rightarrow \lambda^+$, by $F_1(x) = \sup \cup_{y<x} f(y)$. (As $|\cup_{y<x} f(y)| \leqq \lambda \cdot \lambda < \lambda^+$, clearly $F_1(x) \in \lambda^+$ for $x \in \lambda^+$.) Let $S_1 = \{x : y < x$ implies $F_1(y) < x\}$. Clearly $\mathrm{tp}\, S_1 = \lambda^+$ and for $x \in S_1$, $[f(x) \cap S_1] \subseteq \{y : y < x, y \in S_1\}$, so w.l.o.g. $S_1 = \lambda^+$, and

$$f(x) \subseteq x = \{y : y < x\} \text{ for any } x \in \lambda^+. \tag{2}$$

For any sequence $\bar{y} = \langle y_i : i < i_0 \rangle$ ($y_i \in \lambda^+$), define

$$C(\bar{y}) = \{z : (\forall i < i_0)\, [y_i < z \wedge y_i \notin f(z)]\}.$$

Now define another function $F_2 : \lambda^+ \rightarrow \lambda^+$ such that $F_2(x)$ is the least element satisfying the following: For any $i_0 < \mu$, and sequence $\bar{y} = \langle y_i: i < i_0 \rangle$, $y_i < x$, the following holds:

A) If $|C(\bar{y})| < \lambda^+$ then $C(\bar{y}) < F_2(x)$ [that is $z \in C(\bar{y}) \to z < F_2(x)$].

B) If $|C(\bar{y})| = \lambda^+$, then $|\{z: x \leqq z < F_2(x), z \in C(\bar{y})\}| = \lambda$.

C) If $x_1 < x$, $i_1 < \mu$, $\bar{z} = \langle z_i: i < i_1 \rangle$ and

$$[C(\bar{y}) \cap \{v: x_1 \leqq v < x\}] \subseteq \bigcup_{i<i_1} f(z_i)$$

*then* there is such a $\bar{z}$ for which $x < \bar{z} < F_{2i}(x)$.

As $\lambda = \lambda^{\bar{\mu}}$, the number of such $\bar{y}$'s is $\leqq \lambda$, hence $F_2(x) < \lambda^+$. Let $S_2 = \{y: z < y$ implies $F_2(z) < y\}$. Clearly the cardinality of $S_2$ is $\lambda^+$ (in fact it is a closed unbounded subset of $\lambda^+$). Let $\xi$ be an ordinal $\xi < \mu^+$; w.l.o.g. $\xi \geqq \mu$. Order the elements of $\xi + 1$ in a sequence of length $\mu, \xi$ being the first: $\{\alpha: \alpha \leqq \xi\} = \{\alpha_i: i < \mu\}$, $\alpha_0 = \xi$. Choose an increasing sequence of elements of $S_2$ of length $\xi + 2$ $\{x_\beta: \beta < \xi + 2\}$. Now we shall define $y_i$ by induction on $i$ such that:

i) $x_{\alpha_i} \leqq y_i < x_{\alpha_i+1}$

ii) if $j < i$ then $y_j \notin f(y_i)$, $y_i \notin f(y_j)$.

There is no problem in defining $y_0$. Suppose $\mu > i > 0$, and we have defined $y_j$ for $j < i$. By (2) it suffices that the $y_i$ will satisfy

I) $x_{\alpha_i} \leqq y_i < x_{\alpha_i+1}$,

II) $j < i$, $\alpha_j < \alpha_i \to y_j \notin f(y_i)$

III) $j < i$, $\alpha_j > \alpha_i \to y_i \notin f(y_j)$.

If no such $y_i$ exists then

$$[C \cap \{v: x_{\alpha_i} \leqq v < x_{\alpha_i+1}\}] \subseteq \cup\{f(y_j): j < i, \alpha_j > \alpha_i\} \tag{3}$$

where $C = C(\langle y_j: j < i, \alpha_j < \alpha_i \rangle)$.

As $x_{\alpha_i+1} \in S_2$, $F_2(x_{\alpha_i}) < x_{\alpha_i+1}$ and even $F_2(F_2(x_{\alpha_i})) < x_{\alpha_i+1}$.

Hence

$$[C \cap \{v: x_{\alpha_i} \leqq v < F_2(x_{\alpha_i})\}] \subseteq \cup\{f(y_j): j < i, \ \alpha_j > \alpha_i\}.$$

As $y_0 \in C$ by the induction hypothesis (ii), and $F_2(x_{\alpha_i}) < x_{\alpha_{i+1}} \leqq x_{\alpha_0} \leqq y_0$, *clearly* condition (A) of the definition of $F_2$ implies $|C| = \lambda^+$, *hence* condition (B) of the definition of $F_2$ implies

$$|C \cap \{v: x_{\alpha_i} \leqq v < F_2(x_{\alpha_i})\}| = \lambda. \tag{4}$$

So from condition (C) of the definition of $F_2$, and from (3) it follows that there is $\langle y_j^*: j < i, \alpha_j > \alpha_i \rangle$ such that $F_2(x_{\alpha_i}) \leqq y_j^* < F_2(F_2(x_{\alpha_i})) < x_{\alpha_i+1}$ and

(5) $$[C \cap \{v: x_{\alpha_i} \leqq v < F_2(x_{\alpha_i})\}] \subseteq \cup \{f(y_j^*): j \leqq i, \alpha_j > \alpha_i\}.$$

From (3), (4) and (5), remembering $i < \mu$ and $\mathrm{cf}(\lambda) \geqq \mu$ because $\lambda^{\bar{\mu}} = \lambda$, it follows that there are $j, k$, $j < i$, $\alpha_j > \alpha_i$, $k < i$, $\alpha_k > \alpha_i$, such that

(6) $$|f(y_j) \cap f(y_k^*)| = \lambda.$$

But $y_k^* < x_{\alpha_i+1} \leqq x_{\alpha_j} \leqq y_j$ so $y_j \neq y_k^*$, hence (6) contradicts the theorem's hypothesis.

Thus, we can define $y_i$ as needed. By condition (ii) of the inductive definition (of $y_i$) it follows that $\{y_i: i < \mu\}$ is a free subset of $\lambda^+$, and by condition (i) of that definition it follows that $\mathrm{tp}\{y_i: i < \mu\} = \mathrm{tp}\{x_\beta: \beta < \xi + 1\} = \xi + 1$. Thus the theorem is proved.

REMARK. We could have weakened the hypothesis of the theorem to: $(*)_n^\rho$: for every distinct $y_1, \cdots, y_n \in \lambda^+$, $\mathrm{tp}[\cap_{m=1}^n f(y_m)] < \rho$ for any $\rho < \lambda^+ = \aleph_1$ and $n < \omega$; another weakening is $(**)_\kappa$ for every distinct $y_i$ $i < \kappa$, $|\cap_{i<\kappa} f(y_i)| < \lambda$ for any $\kappa$ which satisfies $\aleph_\gamma < \mu \Rightarrow \aleph_\gamma^\kappa < \mathrm{cf}(\lambda)$. The proof is essentially the same.

## 3. On the existence of universal graphs

The term "graph" will mean here an undirected graph with no loop and no multiple edges (but the result can be generalized straightforwardly for those cases). A graph $G$ is an ordered pair $\langle V, E\rangle$, where $V$ is the set of vertices and $E \subseteq [V]^2$ is the set of edges. Here $G_i = \langle V_i, E_i\rangle$ etc. The valency of $a \in V$ is $v(a, G) = |\{b: b \in V, (b,a) \in E\}|$. A function $f$ is an embedding of $G_1$ in $G_2$ if $f: V_1 \to V_2$ is a one-to-one function, and for $a, b \in V_1$, $(a,b) \in E_1$ iff $(f(a), f(b)) \in E_2$. $G_1$ is a (spanned) subgraph of $G_2$ if the identity function on $V_1$ is an embedding. $G_1$ is embeddable in $G_2$ if there is an embedding $f$ of $G_1$ in $G_2$ (that is, if $G_1$ is isomorphic to a subgraph of $G_2$). For a cardinal $\lambda$ let $K_\lambda$ be the class of graphs $G$ such that $|V| \leqq \lambda$ and for $a \in V$, $v(a,G) < \lambda$. $G \in K_\lambda$ is universal in $K_\lambda$ if any $G_1 \in K_\lambda$ is embeddable in $G$. Clearly, if $\lambda > \aleph_0$ is regular, then every component of a graph $G \in K_\lambda$ is of cardinality $< \lambda$. By this, Rado proved that if $\lambda > \aleph_0$ is regular, $\mu < \lambda \to 2^\mu \leqq \lambda$, then $K_\lambda$ has a universal graph. The following theorem shows this is sufficient also for singular $\lambda$.

THEOREM 3.1. *If $\lambda > \aleph_0$ is a strong limit singular cardinal* (i.e. $\mu < \lambda \to 2^\mu < \lambda$) *then in $K_\lambda$ there is a universal graph.*

REMARK. For example $\lambda = \beth_\omega$ satisfies the conditions.

PROOF. As $\lambda$ is singular, let $\lambda = \sum_{i<\kappa} \mu_i$, where $\mu_i < \lambda$, $\kappa < \lambda$, and w.l.o.g $2^{\mu_i} < \mu_{i+1}$; for a limit ordinal $\delta < \kappa$, $\mu_\delta = \sum_{i<\delta} \mu_i$. Denote $\mu[i] = \mu_i$. We shal. define an increasing sequence of graphs $G_i$ and functions $f_i$ for $i \leqq \kappa$ such that:

1) if $j < i$ then $G_j$ is a subgraph of $G_i$

2) $f_i$: $V_i \to \kappa$, and for $j < i$, $f_i$ extends $f_j$

3) for a limit ordinal $\delta \leqq \kappa$, $V_\delta = \cup_{i<\delta} V_i$, $E_\delta = \cup_{i<\delta} E_i$, $f_\delta = \cup_{i<\delta} f_i$

4) $|V_i| = \mu_i$

5) if $i > j > k+3$, $a \in V_j$, $f_i(a) = k$, *then* $\{b: b \in V_i,\ (a,b) \in E_i\} = \{b: b \in V_j,\ (a,b) \in E_j\}$

6) for any $j < i$, any graph $G$, $V = \{x_l: l < \mu_{j+1}\}$, any function $f$ from $V$ into $\kappa$ such that $v(x_l, G) \leqq \mu[f(x_l)]$ and any embedding $g$ into $G_i$ of the subgraph of $G$ spanned by $\{x_l: l < \mu_j\}$ or the empty set such that for any $a$ in the domain of $g$, (Dom, $g$), $f(a) = f_i(g(a))$ and $f(a) < j$ $(a,b) \in E \Rightarrow b \in \mathrm{Dom}\, g$, *there exists* an extension $g'$ of $g$ which is an embedding of $G$ into $G_{i+1}$ such that $f(x_l) = f_{i+1}(g'(x_l))$ for $l < \mu_{j+1}$.

We define $G_i, f_i$ inductively. $G_0$ will be any graph of power $\mu_0$, $f_0$ any function from $V_0$ into $\kappa$. For a limit ordinal $\delta$, $V_\delta = \cup_{j<\delta} V_j$, $E_\delta = \cup_{j<\delta} E_j$, $f_\delta = \cup_{\kappa<\delta} f_i j$ Suppose $G_i, f_i$ have been defined, and we shall define $G_{i+1}, f_{i+1}$. The number of triples $\langle G, g, f\rangle$ such that for some $j < i$:

a) $V = \{x_l: l < \mu_{i+1}\}$

b) $f$ is a function from $V$ into $\kappa$, such that $v(x_l, G) \leqq \mu[f(x_l)]$

c) $g$ is an embedding of the subgraph of $G$ spanned by $\{x_l: l < \mu_j\}$ or the empty set, such that for any $a \in \mathrm{Dom}\, g$, $f(a) = f_i(g(a))$, and $(a,b) \in E$, $f(a) < j \Rightarrow b \in \mathrm{Dom}\, g$

$$\text{is} \leqq \sum_{j<i} 2^{\mu_{j+1}} \cdot \kappa^{\mu_{j+1}} \cdot (\mu_i^{\mu_j} + 1) \leqq 2^{\mu_i} \leqq \mu_{i+1}.$$

Let $\{\langle G^\alpha, g^\alpha, f^\alpha\rangle : \alpha < \mu_{i+1}\}$ be a list of all such triples (possibly with repetitions), and $G^\alpha = \{x_l: l < \mu_{k(\alpha)}\}$.

Let $V_{i+1} = V_i \cup \{\langle i+1, \alpha, l\rangle : \alpha < \mu_{i+1}, l < \mu_{k(\alpha)}, x_l$ is not in the domain of $g^\alpha\}$, $E_{i+1} = E_i \cup \{(a, \langle i+1, \alpha, l\rangle) : a \in V_i,\ \langle i+1, \alpha, l\rangle \in V_{i+1}$, for some $l'$ $a = g^\alpha(x_{l'})$ and $(x_l, x_{l'}) \in E^\alpha\} \cup \{(\langle i+1, \alpha, l\rangle, \langle i+1, \alpha, l'\rangle) : \langle i+1, \alpha, l\rangle, \langle i+1, \alpha, l'\rangle \in V_{i+1}, (x_l, x_{l'}) \in E^\alpha\}$,

$$f_{i+1}(a) = \begin{cases} f_i(a) & a \in V_i, \\ f^\alpha(x_l) & a = \langle i+1, \alpha, l\rangle. \end{cases}$$

It is easy to check that all conditions are satisfied. Now $|V_\kappa| = \lambda$ by (3), (4) and if

$a \in V_\kappa$ then $a \in V_i$ for some $i$; hence, letting $k = \max(i, f_i(a)) < \kappa$, we get by condition (5)

$$v(a, G_\kappa) = |\{b : b \in V_\kappa, (a,b) \in E_\kappa\}| = |\{b : b \in V_{k+4}, (a,b) \in E_{k+4}\}|$$
$$\leqq |V_{k+4}| = \mu_{k+4} < \lambda.$$

So $G_\kappa \in K_\lambda$. We shall show that $G_\kappa$ is universal, thus proving the theorem. Let $G^* \in K_\lambda$. We can easily define an increasing sequence of subgraphs $G_i^*$ such that

A) $|V_i^*| = \mu_i$

B) $a \in V_i^*$, $(a,b) \in E$, $v(a, G^*) < \mu_i$ implies $b \in V_i^*$

C) if $a \in V_i^*$, $v(a, G^*) \geqq \mu_i$ then $v(a, G_i^*) = \mu_i$

D) for a limit ordinal $\delta \leqq \kappa$, $V_\delta^* = \cup_{i<\delta} V_i^*$.

Let $f : V^* \to \kappa$ be such that $\mu[f(a)] \geqq v(a, G^*)$. Now w.l.o.g. $V_i^* = \{x_l : l < \mu_i\}$ (otherwise replace $G^*$ by a suitable graph isomorphic to it). Now we can define by condition (6) an increasing sequence of embeddings $g_i$ of $G_i^*$ into $G_{i+2}$ such that if $a \in V_i^*$ then $f_i(g(a)) = f(a)$. Then clearly $g_\kappa = \cup_{<\kappa} g_i$ is the required embedding.

DEFINITION: A graph $G$ is strongly universal in $K_\lambda$ if $G \in K_\lambda$ and every $G^* \in K_\lambda$ can be embedded into $G$ by a valency-preserving $g$ (that is, $g : V^* \to V$ is an embedding, and $v(a, G^*) = v(g(a), G)$ for any $a \in G^*$).

THFOREM 3.2 (G.C.H.). *If $\lambda$ is singular, $\lambda = \aleph_\alpha > \kappa = |\alpha| + \aleph_0$, then in $K_\lambda$ there is a strongly universal graph.*

REMARKS.

1) Instead of G.C.H., we can assume there is $\mu < \lambda$ such that for any $\kappa$, $\mu < \kappa < \lambda$, $\kappa^+ = 2^\kappa$.

2) For regular $\lambda$, assuming G.C.H., there is a strongly universal graph in $K_\lambda$—Rado's proof provides it in fact. We just have to take a graph $G$ such that for a connected $G^* \in K_\lambda$ there are $\lambda$ components of $G$ isomorphic to $G^*$.

PROOF. For any graph $G^* \in K_\lambda$, we can find $G_i^*$ $i \leqq \lambda$ such that:

1) for $j < i$, $G_j^*$ is a subgraph of $G_i^*$,

2) if $\delta \leqq \lambda$ is a limit ordinal then $\cup_{i<\delta} V_i^* = V_\delta$ and $G_\lambda^* = G^*$,

3) $|V_i^*| = |i| + \kappa$ and if $a \in V_i^*$, $b \in V^*$, $(a,b) \in E$ and $v(a, G^*) \leqq |V_i^*|$, then $b \in V_i^*$,

4) If $a \in V_i^*$, $v(a, G^*) \geqq |V_i^*|$ then $v(a, G_i^*) = |V_i^*|$.

This can be done easily. The only difficulty is in condition (3) for a limit ordinal $i$.

For each $a \in V^*$, let $C(a, G^*) = \{b: b \in V^*, (a,b) \in E^*\}$, let $<_a$ be a well ordering of $C(a, G^*)$ of order-type $|C(a, G^*)|$, and when defining inductively $G_i^*$, demand in addition to (1)–(4)

5) if $a \in V_i^*$, the first $|V_i^*|$ members of $C(a, G^*)$ belong to $V_i^*$ (first by $<_a$).

Now we can define inductively $G_i, f_i$ for $i \leqq \lambda$ such that

A) for $j < i$, $G_j$ is a subgraph of $G_i$, $f_i: V_i \to \alpha$ and $f_i$ extends $f_j$,

B) if $\delta \leqq \lambda$ is a limit ordinal then $\cup_{i<\delta} V_i = V_\delta$, and $G = \cup_{i<\lambda} G_i$,

C) $|V_i| = |i| + \kappa$,

D) for $a \in V_j$, $j < i$, $\aleph_{f_j(a)} \leqq |V_j|$ iff for no $b \in V_i - V_j$ $(a,b) \in E_i$,

E) if $a \in V_i$, $\aleph_{f_i(a)} \leqq |V_i|$, *then* $v(a, G_i) = \aleph_{f_i(a)}$,

F) if $G^*$ is a graph, $|V^*| = \kappa$, $f^*: V^* \to \kappa$ and for $a \in V^*$, $v(a, G^*) = \min[\kappa, \aleph_{f^*(a)}]$, *then* there is a $j < \kappa^+$ and an embedding $g$ of $G^*$ into $G_j$ such that for $a \in V^*$ $f_j(g(a)) = f^*(a)$,

G) Suppose $G_1^*$ is a subgraph of $G_2^*$, $f^*: V_2^* \to \kappa$; for $a \in V_2^*$, $v(a, G_2^*) = \min[|V^*|, \aleph_{f^*(a)}]$, for any $a \in G_1^*$ $v(a, G_1^*) = v(a, G_2^*)$ and $\aleph_{f^*(a)} \leqq |V_2^*|$ implies $(\forall b \in V_2^*)$ $[(a,b) \in E_2^* \to b \in V_1^*]$. Suppose in addition that $i < \lambda$, $|V_1^*| = |V_2^*| = |i| + \kappa$, $g_1$ is an embedding of $G_1^*$ into $G_i$, and for any $a \in V_1^*, f_i(g_1(a)) = f^*(a)$. Then there is $j$, $i \leqq j < (|i| + \kappa)^+$, and an extension $g_2$ of $g_1$ which is an embedding of $G_2^*$ into $G_j$, and for any $a \in V_2^*$, $f_j(g_2(a)) = f^*(a)$.

This is possible because for each $i$, (F) and (G) produce $2^{|i|+\kappa} = (|i| + \kappa)^+$ demands, which we scattered among the $j + 1 < (|i| + \kappa)^+$; so defining $G_{j+1}$, $f_{j+1}$, we have $\leqq |j| + \kappa$ demands for extending embeddings. Now given a $G^*$, as mentioned before, we can define $G_i^*$ satisfying (1)–(4), and let $f: V^* \to \kappa$ be such that $v(a, G^*) = \aleph_{f(a)}$. Then by (F), we embed $G_0^*$, and by (G), we extend the embedding inductively.

THEOREM 3.3. *If $\aleph_\alpha = \alpha$, $\aleph_\alpha$ singular then there is no strongly universal graph in $K_{\aleph_\alpha}$.*

PROOF. Suppose $G$ is such a graph. Let $\kappa$ be the cofinality of $\aleph_\alpha$, $\aleph_\alpha = \sum_{i<\kappa} \lambda_i$, $\lambda_i < \aleph_\alpha$. Let $\{x_i: i < i_0 \leqq \aleph_\alpha\}$ be the set of vertices of $G$ of valency $\kappa$. Foı $i < \kappa$, let $S_i = \{v(y, G)$: there is $j < \lambda_i$ such that $(y, x_j) \in E\}$. As $v(x_i, G) = \kappa$, for each $j$ there are $\kappa$ $y$'s for which $(y, x_i) \in E$. Hence $|S_i| \leqq \kappa \cdot \lambda_j < \aleph_\alpha$. Choose a cardinal $\mu_i < \aleph_\alpha$, $\mu_i \notin S_i$. Define a graph $G^*$:

$$V^* = \{x\} \cup \{y_j: j < \kappa\} \cup \{z_{j,i}: j < \kappa, i < \mu_j\}$$

$$E^* = \{(x, y_j): j < \kappa\} \cup \{(y_j, z_{j,i}): j < \kappa, i < \mu_j\}.$$

Suppose $g$ is a strong embedding of $G^*$ into $G$. Then $v(g(x), G) = \kappa$, hence there is $k < i_0$ such that $g(x) = x_k$. Let $k < \lambda_j$, then $(g(y_j), x_k) \in E$, so $v(g(y_j), G) \in S_j$. But $v(y_j, G^*) = \mu_j \notin S_j$, a contradiction.

REMARK. Another universal graph which exists is as follows. Let $T$ be a set of finite graphs, $K_T$ be the class of graphs $G$, such that every finite spanned-subgraph of $G$ is isomorphic to a member of $K$. Then (G.C.H.) for $\aleph_\alpha \geqq \aleph_\beta > \aleph_0$, among the graphs in $K_T$ of cardinality $\leqq \aleph_\alpha$ and chromatic number $\leqq \aleph_\beta$, there is a universal one.

## 4. On the existence of Jonsson algebras

An algebra, here always with countably many functions, is a Jonsson algebra if every proper subalgebra has a smaller cardinality.

For cardinals $\lambda, \mu, \kappa$ and natural number $n$, $\lambda \to [\mu]^n_\kappa$ means that for every function $f: [\lambda]^n \to \kappa$ there is $S \subseteq \lambda$ and $i_0 < \kappa$ such that $|S| = \mu$ and for no distinct $a_1, \cdots, a_n \in S$, $f(a_1, \cdots, a_n) = i_0$. The negation of this statement is $\lambda \not\to [\mu]^n_\kappa$. As mentioned in Erdös and Hajnal [2], $\lambda \not\to [\lambda]^n_\lambda$ implies there is a Jonsson algebra of cardinality $\lambda$ (if $f$ is a function for which the definition fails then $(\lambda, f)$ is the desired algebra).

THEOREM 4.1. *If $2^{\aleph_\alpha} \leqq \aleph_{\alpha+n}$ then there is a Jonsson algebra of cardinality, $\aleph_{\alpha+n}$.*

By the last remark it suffices to prove

THEOREM 4.2. *If $2^{\aleph_\alpha} \leqq \aleph_{\alpha+n}$ then $\aleph_{\alpha+n} \not\to [\aleph_{\alpha+n}]^{n+1}_{\aleph_{\alpha+n}}$.*

Proof. We shall define by induction on $m \leqq n$ functions $F_m$ with $m$ places, with domain $\aleph_{\alpha+n}$, and range the family of subsets of $\aleph_{\alpha+n}$ of cardinality $\leqq \aleph_{\alpha+n-m}$. For $m = 0$, $F_m(\ ) = \aleph_{\alpha+n}$.

Suppose $m < n$ and $F_m$ has been defined. Then for every $a_1, \cdots, a_m \in \aleph_{\alpha+n}$, choose a well ordering $<_{\bar a}(\bar a = \langle a_1, \cdots, a_m \rangle)$ of $F(a_1, \cdots, a_m)$ of order type $|F(a_1, \cdots, a_m)|$. Now define

$$F_{m+1}(a_1, \cdots, a_{m+1}) = \begin{cases} \{x : x \in F(\bar a),\ x <_{\bar a} a_{m+1}\} & \text{if } a_{m+1} \in F(\bar a) \\ & \text{where } \bar a = \langle a_1, \cdots, a_m \rangle, \\ \emptyset & \text{otherwise.} \end{cases}$$

LEMMA 4.3. *If $A \subseteq \aleph_{\alpha+n}$, $|A| = \aleph_{\alpha+n}$, $B \subseteq \aleph_{\alpha+n}$, $|B| = \aleph_\alpha$ then there are $a_1, \cdots, a_n \in A$ such that $B \subseteq F_n(a_1, \cdots, a_n)$ and $|F_n(a_1, \cdots, a_n)| = \aleph_\alpha$.*

PROOF OF THE LEMMA. Define by induction on $m$, elements $a_m$ such that

1) $a_m \in A$

2) $B \subseteq F_m(a_1, \cdots, a_m)$

3) $\aleph_{\alpha+n-m} = |F_m(a_1, \cdots, a_m)| = |A \cap F_m(a_1, \cdots, a_m)|$.

Suppose we have defined $a_l$ for $l \leqq m$ and $0 \leqq m < n$, and the induction conditions are satisfied (for $m = 0$, (2) and (3) are satisfied by the assumptions of the lemma and definition of $F_0$). We shall define $a_{m+1}$. Let $\bar{a} = \langle a_1, \cdots, a_m \rangle$. Since $B \subset F_m(\bar{a})$, $|B| = \aleph_\alpha < \aleph_{\alpha+n-m} = |F_m(\bar{a})|$ and $<_{\bar{a}}$ is a well ordering of $F_m(\bar{a})$ of order-type $\aleph_{\alpha+n-m}$ which, as a successor cardinal, is regular, it is clear that $B$ is bounded in $F_m(\bar{a})$ by the order $<_{\bar{a}}$, by an element we shall call $b_1$. Similarly, as $|A \cap F_m(\bar{a})| = \aleph_{\alpha+n-m}$, there is a $b_2 \in F_m(\bar{a})$, $b_1 <_{\bar{a}} b_2$, such that $|A \cap \{b: b \in F_m(\bar{a}), b <_{\bar{a}} b_2\}| = \aleph_{\alpha+n-m-1}$. Choose $a_{m+1}$ as any element of $F_m(\bar{a}) \cap A$ which is bigger (by $<_{\bar{a}}$) than $b_2$. (There exists one since $|F_m(\bar{a})| = |F_m(\bar{a}) \cap A| = \aleph_{\alpha+n-m}$, $<_{\bar{a}}$ has order type $\omega_{\alpha+n-m}$, and $A$ is unbounded in $F_m(\bar{a})$). Clearly $a_{m+1}$ satisfies the induction conditions.

Let us return to the proof of the theorem. By our assumptions, $\aleph_{\alpha+n}^{\aleph_\alpha} = \aleph_{\alpha+n}$, hence the number of subsets of $\aleph_{\alpha+n}$ of cardinality $\aleph_\alpha$ is $\aleph_{\alpha+n}$. Let

$$\{B_i : i < \omega_{\alpha+n}\} = \{B : B \subseteq \aleph_{\alpha+n}, |B| = \aleph_\alpha\}.$$

Now the following observation is well known (it is proved using $\aleph_\alpha^2 = \aleph_\alpha$).

(*) For an infinite cardinality $\aleph_\alpha$, a set $A$ of cardinality $\aleph_\alpha$, and a family $\{A_i : i \in I\}$ of subsets of $A$, $|A_i| = \aleph_\alpha$, $|I| = \aleph_\alpha$, we can find $B_i \subseteq A_i$, $|B_i| = \aleph_\alpha$, $i \neq j \to B_i \cap B_j = \emptyset$. Hence we can find a function of $f: A \to A$, such that for any $i \in I$, $\{f(a): a \in A_i\} = A$.

For any $\bar{a} = \langle a_1, \cdots, a_n \rangle$, $a_i \in \aleph_\alpha$, $|F_n(\bar{a})| = \aleph_\alpha$, we can define a one-place function $f_{\bar{a}}: F_n(\bar{a}) \to F_n(\bar{a})$ such that if $i \in F_n(\bar{a})$, $B_i \subseteq F_n(\bar{a})$ (or even $|B_i \cap F_n(\bar{a})| = \aleph_\alpha$) then $\{f_{\bar{a}}(b): b \in B_i\} = F_n(\bar{a})$. If $|F_n(\bar{a})| < \aleph_\alpha$, define $f_{\bar{a}}$ arbitrarily. Now we define the function $g$ which will show the truth of Theorem 4.2:

$$g(a_1, \cdots, a_n, a_{n+1}) = f_{\langle a_1, \cdots, a_n \rangle}(a_{n+1}).$$

We should show that for any $A \subseteq \aleph_{\alpha+n}$, $|A| = \aleph_{\alpha+n}$, $\{g(a_1, \cdots, a_{n+1}): a_i \in A\} = \aleph_{\alpha+n}$. Let $x \in \aleph_{\alpha+n}$, and we shall find $a_1, \cdots, a_{n+1} \in A$, $g(a_1, \cdots, a_{n+1}) = x$. Choose a subset $A^*$ of $A$ of cardinality $\aleph_\alpha$; thus, by the definition of the $B_i$'s there is $i < \omega_{\alpha+n}$ such that $A^* = B_i$. Let $B = B_i \cup \{i, x\}$; clearly $|B| = \aleph_\alpha$ and $B \subset \aleph_{\alpha+n}$, hence by Lemma 4.2, there are $a_1, \cdots, a_n \in A$ such that $|F_n(\bar{a})| = \aleph_\alpha$, $B \subseteq F_n(\bar{a})$ where $\bar{a} = \langle a_1, \cdots, a_n \rangle$.

By the definition of $f_{\bar{a}}$, since $i \in F_n(\bar{a})$, $B_i \subseteq B \subseteq F_n(\bar{a})$, $x \in B \subseteq F_n(\bar{a})$, there is

$a_{n+1} \in B_i = A^* \subseteq A$ such that $f_{\bar a}(a_{n+1}) = x$; hence, $g(a_1, \cdots, a_n, a_{n+1}) = f_{\bar a}(a_{n+1}) = x$. Thus, the theorem is proved.

## 5. A combinatorial lemma

Here a colouring of $S$ by $\kappa$ colours is a function $C$ from $[S]^2$ into a set of cardinality $\kappa$, which is w.l.o.g. $\kappa = \{i: i < \kappa\}$.

LEMMA 5.1. *Let* $\mathrm{tp}\langle S, <\rangle = \omega_{\alpha+1}$, $\aleph_\beta > \kappa^{\bar\mu}$, $\aleph_\alpha = \aleph_\alpha^{\tilde{\aleph}_\beta}$ *and* $\aleph_\beta$, $\mu$ *are regular. Assume in addition* $|I| \leqq \kappa$ *and for each* $t \in I$, $C_t$ *is a colouring of S by* $\leqq \kappa$ *colours. Then there is a set* $S_1 \subseteq S$, $|S_1| = \aleph_{\alpha+1}$ *(hence* $\mathrm{tp}(S_1) = \omega_{\alpha+1}$*) such that for every* $J \subseteq I$, $|J| < \mu$, *and* $a_1, a_2 \in S_1$, $a_3 \in S$, $a_1 < a_2$, *there is an increasing sequence* $\langle b_i: 0 < i \leqq \omega_\beta\rangle$ *of elements of S*, $a_3 < b_0$ *which satisfies*

A) *for every* $t \in J$, $i < j \leqq \omega_\beta$, $C_t(b_i, b_j) = C_t(a_1, a_2)$

B) *for every* $t \in I$ *and* $i < j < k \leqq \omega_\beta$, $C_t(b_i, b_j) = C_t(b_i, b_k)$.

REMARK 5.1. If $S = \omega_{\alpha+1}$, then $S_1$ is stationary.

REMARK 5.2. Assuming G.C.H., $\alpha = \beta$, $\mu = \kappa^+$, the demand on the cardinals is "$\aleph_\alpha$ regular, $\kappa^+ < \aleph_\alpha$".

PROOF. Without loss of generality, $S = \omega_{\alpha+1}$; it is clear that $\kappa^{\bar\mu} < \aleph_\beta \leqq \aleph_\alpha$. Define for any $a \in S$, $A \subseteq S$, $T(a, A) = \{\langle b, i, t\rangle: t \in I,\ b \in A, \text{ and } i = C_t(a, b)\}$. Now for any $a \in S$ of cofinality $\geqq \aleph_\beta$, any $J \subseteq I$, $|J| < \mu$ and any function $g: J \to \kappa$, consider the following condition:

> $[*(a, J, g)]$ for any set $A \subseteq \{b: b < a\}$, $|A| < \aleph_\beta$, there is $a^1$, $A < a^1 < a$ such that $T(a, A) = T(a^1, A)$ and for any $t \in J$, $C_t(a^1, a) = g(t)$.

If $[*(a, J, g)]$ fails, let $A(a, J, g)$ be a set contradicting it and if $[*(a, J, g)]$ holds, let $A(a, J, g) = \varnothing$. Let

$$A(a) = \cup\{A(a, J, g): J \subseteq I,\ |J| < \mu,\ g: J \to \kappa\}.$$

From the assumptions on the cardinals it follows that $|A(a)| < \aleph_\beta$. Hence for any $a \in S$ of cofinality $\geqq \aleph_\beta$, $A(a)$ is bounded below $a$, so choose $f(a)$, $A(a) < f(a) < a$. Since $S^1 = \{a: a \in \omega_{\alpha+1}, \mathrm{cf}(a) \geqq \aleph_\beta\}$ is a stationary subset of $\omega_{\alpha+1}$ and $f(a) < a$ for $a \in S^1$, by a well-known result of Fodor [16] $f$ is constant over some stationary subset $S^2$ of $S^1$, and let $f(a) = b_0$ for any $a \in S^2$. Since the number of subsets of $\{b: b < b_0\}$ of cardinality $< \aleph_\beta$ is $\leqq \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$, the equivalence relation $A(x) = A(y)$ partitions $S^2$ into $\aleph_\alpha$ equivalence classes, so at least one of them

$S^3 \subseteq S^2$ is stationary. Let $A(a) = A_0$ for $a \in S^3$. Similarly there is a stationary $S^4 \subseteq S^3$ such that $T(a, A_0) = T_0$ for any $a \in S^4$.

Let $a_0$ be the first element of $S^4$ such that $a_0 > A_0$ and for any $g: I \to \kappa$, if there are $b > a > a_0$ such that for any $t \in I$ $C_t(a,b) = g(t)$ then there are such $b, a$ with arbitrarily large $a$ (since the number of functions

$$g: I \to \kappa \text{ is } \leqq \kappa^{|I|} = \kappa^\kappa \leqq \aleph_\alpha^{\tilde{\aleph}_\beta} = \aleph_\alpha,$$

there is such $a_0$). Let $S_1 = \{a : a \in S^4,\ a_0 < a\}$.

Let us show that $S_1$ satisfies the conclusion of the theorem. Clearly $|S_1| = \aleph_{\alpha+1}$. Now suppose $a_1 < a_2$, $a_1, a_2 \in S_1$, $a_3 \in S$, $J \subseteq I$, $|J| < \mu$. Define $g: J \to \kappa$ by $g(t) = C_t(a_1, a_2)$. By the definition of $S_1$ and $a_0$, there are $a^1, a^2 \in S^4$, $a_0 < a^1 < a^2$, $a_3 < a^1 < a^2$ (hence $a^1, a^2 \in S_1$) such that for any $t \in I$, $C_t(a^1, a^2) = C_t(a_1, a_2)$. Clearly $f(a^1) = f(a^2) = b_0$, $A(a^1) = A(a^2) = A_0$, $T(a^1, A_0) = T(a^2, A_0) = T_0$ and $A(a^2, J, g) \subseteq A(a^2) = A_0$. But $A_0 < a^1 < a^2$, $T(a^1, A_0) = T(a^2, A_0)$ and for each $t \in J$, $C_t(a^1, a^2) = g(t)$. Hence by the definition of $A(a^2, J, g)$ the condition $[*(a^2, J, g)]$ should hold. Define $b_{\omega_\beta} = a^2$, and we now define $b_i$, $0 < i < \omega_\beta$ by induction on $i$. If $i < \omega_\beta$ and $b_j < a^2$ is defined for each $0 < j < i$, choose $b_i$ as the first element satisfying $B_i = [A_0 \cup \{a_3\} \cup \{b_j : j < i\}] < b_i < a^2$, $T(a^2, B_i) = T(b_i, B_i)$ and for every $t \in J$, $C_t(b_i, a^2) = g(t)$. This can be done as $[*(a^2, J, g)]$ holds. It is easy to see that $\langle b_i : 0 < i \leqq \omega_\beta \rangle$ satisfies the demands in the conclusion.

REMARK 5.3. If $a_3 < a_2$, by the proof we can choose $b_{\omega_\beta} = a_2$.

REMARK 5.4. By small changes, we can define a decreasing sequence of stationary subsets of $\omega_{\alpha+1}$: $S_i$ $i < \omega_\beta$ such that for $i < j$ the pair $(S_i, S_j)$ satisfies what the conclusion says on $(S, S_1)$.

REMARK 5.5. It may be interesting to look at the following associated colouring of $\omega_{\alpha+1}$:

> $C(a,b) = \{(g_1, g_2)$: $g_1 g_2$ are functions from $I$ to $\kappa$ and for every $A < a < b$ $|A| < \aleph_\beta$ there is $a^*$, $A < a^* < a$ such that $g_1(t) = C_t(a^*, a)$, $g_2(t) = C_t(a^*, b)$ we can and $T(a, A) = T(a^*, A)\}$.

REMARK 5.6. Extending the definition of $[*(a, J, g)]$ to any $J \subseteq I$, if $2^{2^\kappa} \leqq \aleph_\alpha$ we can add to the conclusion of the lemma:

C) *for any* $a, b \in S_1$ *and any* $g: I \to \kappa$, $[*(a, I, g)]$ *and* $[*(b, I, g)]$ *are equivalent.*

This is true because $|\{g : g: I \to \kappa\}| \leqq 2^\kappa$. We can add this to the construction in Remark 5.4 and in addition demand that for any $a \in S_{i+1}$, the set of $(I, g)$ $(J \subseteq I,\ |J| < \mu,\ g: J \to \kappa)$ for which $[*(a, J, g)]$ is satisfied relative to $S_i$ is independent of $a$ and $i$.

## 6. A partition theorem

We shall deal here with an arbitrary 2-colouring of $\omega_{\alpha+1}$ by two colours, which will be red ( = 0) and blue ( = 1). Instead of $C(a, b) = 0$, we say $(a, b)$ is red (by $C$) or $C(a, b)$ is red. For simplicity we shall assume G.C.H. $\alpha \to (\beta)^n_\kappa$ means that for any function $f: [\alpha]^n \to \kappa$ there is $A \subseteq \alpha$, $\operatorname{tp}(A) = \beta$, such that $f$ is constant over $\lambda$.

THEOREM 6.1. (G.C.H.) *If $\aleph_\alpha$ is regular $> \aleph_0$, $|\xi|^+ < \aleph_\alpha$, then $\omega_{\alpha+1} \to (\omega_\alpha + \xi)^2_2$.*

PROOF. Let $\aleph_\gamma = |\xi|^+$, so clearly $\aleph_\gamma$ is a regular cardinal. Let $C$ be a colouring of $\omega_{\alpha+1}$ by red and blue (that is $C: [\omega_{\alpha+1}]^2 \to \{0, 1\}$). A subset $X$ of $\omega_{\alpha+1}$ is red (blue) if all the pairs from it are red (blue). By Lemma 5.1 we can assume w.l.o.g. that for any $a \in \omega_{\alpha+1}$ there is a red set $A > a$ such that $\operatorname{tp}(A) = \omega_\alpha + 1$ (otherwise $\omega_{\alpha+1}$ has a blue subset of order-type $\omega_{\alpha+1}$). Thus we can choose red sets $X_i \subseteq \omega_{\alpha+1}$, $\operatorname{tp}(X_i) = \omega_\gamma$ for $i < \omega_{\alpha+1}$ such that $X_i < X_j$ for $i < j < \omega_{\alpha+1}$. Let $S = \{X_i: i < \omega_{\alpha+1}\}$, (the order $<$ on $S$ is already defined) and $\operatorname{tp}\langle S, < \rangle = \omega_{\alpha+1}$. Let $X(k)$ be the $k$'th element of $X$. We define now several colourings of $S$:

α) $$\text{for any } k, l < \omega_\gamma,\ C_{k,l}(X_i, X_j) = C(X_i(k), X_j(l))$$

β) $$C^*(X_i, X_j) = \begin{cases} 0 \text{ if} \left|\{l: C_{l,l}(X_i, X_j) = 0\}\right| = \aleph_\gamma \\ 1 \text{ otherwise} \end{cases}$$

γ) $$C^{**}(X_i, X_j) = \sup\{k: C_{k,k}(X_i, X_j) = 0\}.$$

Now by Lemma 5.1 we can define by induction on $n$ a decreasing sequence of subsets $S_n$ of $\omega_{\alpha+1}$ of cardinality $\aleph_{\alpha+1}$ such that $S_{n+1}$ is related to $S_n$ as $S_1$ is related to $S$ in the conclusion of 5.1 (where the colourings are the colourings listed above and $\aleph_\gamma, \aleph_\gamma, \aleph_{\gamma+1}, \aleph_\alpha$ stand respectively for $\kappa, \mu, \aleph_\beta, \aleph_\alpha$). The proof is now divided into three cases.

*Case I.* In $S_1$, there are $Y_1 < Y_2$ and $l < \omega_\gamma$ such that $C_{l,l}(Y_1, Y_2)$ is red and

$$\operatorname{tp}\{k: k < \omega_\gamma,\ C_{l,k}(Y_1, Y_2) \text{ is red}\} \geqq \xi.$$

We choose an increasing sequence $k(\rho) < \omega_\gamma$ for $\rho < \xi$ such that $C_{l,k(\rho)}(Y_1, Y_2)$ is red. By the conclusion of Lemma 5.1 (with $\{C_t: t \in J\}$ corresponding to

$\{C_{l,l}, C_{l,k(\rho)}: \rho < \xi\}$) there is an increasing sequence $Z_i$, $i \leqq \omega_\alpha$ in $S$ such that for any $i < j \leqq \omega_\alpha$ and $\rho < \xi$, $C_{l,l}(Z_i, Z_j)$, $C_{l,k(\rho)}(Z_i, Z_j)$ are red. It is easy to check that

$$\{Z_i(l): i < \omega_\alpha\} \cup \{Z_{\omega_\alpha}(k): k = k(\rho); \rho < \xi\}$$

is a red subset of $\omega_{\alpha+1}$ of type $\omega_\alpha + \xi$.

*Case II.* Case I never happens and there are $\beta < \omega_\gamma$ and $Y_1 < Y_2$ in $S_2$ such that (i) $C_{\beta,\beta}(Y_1, Y_2)$ is blue and (ii) $C^*(Y_1, Y_2) = 0$.

Choose $\rho$ $\xi \leqq \rho < \omega_\gamma$ such that $\rho \to (\rho)^1_2$ (clearly there is such a $\rho$; in fact, $\rho = \xi \cdot \omega$ suffices (see [9])). Using again Lemma 5.1, we can find in $S_1$ increasing sequences $\langle Y^i: i < \omega_\alpha \rangle$ and $\langle Z^j: j < \rho \rangle$ such that:

1) $Y^i < Z^j$ for any $i < \omega_\alpha$, $j < \rho$
2) for any $i < j < \omega_\alpha$, $C_{\beta,\beta}(Y^i, Y^j)$ is blue and $C^*(Y^i, Y^j) = 0$
3) for any $i < j < \rho$, $C_{\beta,\beta}(Z^i, Z^j)$ is blue and $C^*(Z^i, Z^j) = 0$
4) for any $i^1 < i^2 < i^3 < \rho$, and $k, l < \omega_\gamma$, $C_{k,l}(Z^{i^1}, Z^{i^2}) = C_{k,l}(Z^{i^1}, Z^{i^3})$.

We can define an equivalence relation $\sim$ over $\{Y^i: i < \omega_\alpha\}$ by $Y^i \sim Y^j$ if for every $k < \rho$, $C_{\beta,\beta}(Y^i, Z^k) = C_{\beta,\beta}(Y^j, Z^k)$. Clearly there are $\leqq 2^{|\rho|} \leqq \aleph_\gamma < \aleph_\alpha$ equivalence classes, hence at least one of them has cardinality $\aleph_\alpha$, hence w.l.o.g. there is only one equivalence class (otherwise we replace the $Y^i$'s by a subsequence of them). Using $\rho \to (\rho)^1_2$, we can similarly replace the $Z^j$ by a subsequence of the same length so that (w.l.o.g.)

(A) for any $i, j < \omega_\alpha$, $k, l < \rho$, $C_{\beta,\beta}(Y^i, Z^k) = C_{\beta,\beta}(Y^j, Z^l)$.

If $C_{\beta,\beta}(Y^0, Z^0)$ is blue then clearly

$$\{Y^i(\beta): i < \omega_\alpha\} \cup \{Z^j(\beta): j < \rho\}$$

is a blue subset of $\omega_{\alpha+1}$ of order-type $\omega_\alpha + \rho \geqq \omega_\alpha + \xi$, so we are finished. We can assume

(B) for any $i < \omega_\alpha$, $k < \rho$, $C_{\beta,\beta}(Y^i, Z^k)$ is red.

As Case I never occurs and $Y^i, Z^k \in S_1$, for every $i < \omega_\alpha$, $j < \rho$,

$$\mathrm{tp}\{k: k < \omega_\gamma,\ C_{\beta,k}(Y^i, Z^j) \text{ is red}\} < \xi.$$

Hence there is $l(i,j) < \omega_\gamma$ such that $l(i,j) \leqq k < \omega_\gamma$ implies $C_{\beta,k}(Y^i, Z^j)$ is blue.

Since $\aleph_\gamma$ is regular and $\rho < \omega_\gamma$, we have that: $l(i) = \sup_{j<\rho} l(i,j) < \omega_\gamma$. As $l$ is a function from $\omega_\alpha$ into $\omega_\gamma < \omega_\alpha$ and $\omega_\alpha$ is regular, clearly by replacing $\langle Y^i: i < \omega_\alpha \rangle$ again by a subsequence of the same length we get that $l$ is constant, i.e. $l(i) = l_0$. That is

(C) for any $i < \omega_\alpha, j < \rho$ and $l_0 \leqq k < \omega_\gamma$, $C_{\beta,k}(Y^i, Z^j)$ is blue.

Now we define by induction on $j < \rho$ ordinals $k(j)$ such that

i) $l_0 < k(j) < \omega_\gamma$

ii) if $i < j$ then $C_{k(i),k(j)}(Z^i, Z^j)$ is blue

iii) $C_{k(j),k(j)}(Z^j, Z^{j+1})$ is red.

It is easy to see by (C), (i), (ii) that if we succeed in the definition then

$$\{Y^i(\beta): i < \omega_\alpha\} \cup \{Z^j(k(j)): j < \rho\}$$

is a blue subset of $\omega_{\alpha+1}$. Suppose we have defined $k(i)$ for every $i < j$, and $j < \rho$ We shall define $k(j)$. By condition (4) of the definition of the $Y^i$'s and $Z^i$'s and the induction hypothesis (iii), for any $i < j$, $C_{k(i),k(i)}$ $(Z^i, Z^j)$ is red. As Case I never happens and $Z^i, Z^j \in S_1$, clearly $\mathrm{tp}\, R^i_j < \xi$ where

$$R^i_j = \{k: k < \omega_\gamma,\ C_{k(i),k}(Z^i, Z^j) \text{ is red}\}.$$

As $j < \rho < \omega_\gamma$, $\xi < \omega_\gamma$, $\aleph_\gamma$ is regular, clearly $R^i_j$ is a bounded subset of $\omega_\gamma$, anp even $\bigcup_{i<j} R^i_j \cup \{l_0\}$ is a bounded subset of $\omega_\gamma$, say by $l^j$. But by condition (3) from the definition of the $Y^i$'s and $Z^i$'s, $C^*(Z^j, Z^{j+1}) = 0$, that is for $\aleph_\gamma$ $k$'s, $C_{k,k}(Z^j, Z^{j+1})$ is red. So there is $k(j)$, $l^j < k(j) < \omega_\gamma$, $C_{k(j),k(j)}$ $(Z^j, Z^{j+1})$ is red. This $k(j)$ clearly satisfies the induction condition, hence we define the $k(j)$'s and so prove the theorem also in the second case.

*Case III.* Case I and II never happen.

We can find $Y_1 < Y_2$ in $S_3$ such that $C_{0,0}(Y_1, Y_2)$ is blue (otherwise $\{Y(0): Y \in S_3\}$ is a red subset of $\omega_{\alpha+1}$ of cardinality $\aleph_{\alpha+1}$). As Case II never happened and $S_3 \subseteq S_2$, clearly $k_0 = C^{**}$ $(Y_1, Y_2) < \omega_\gamma$. Now we can proceed just as in Case II up to (C) with $S_3$ instead of $S_2$, and $S_2$ instead of $S_1$. In the definition of the $Y^i$'s and $Z^j$'s in (2), (3), instead of $C^*(Y^i, Y^j) = 0$, $C^*(Z^i, Z^j) = 0$, we demand $k_0 = C^{**}(Y^i, Y^j)$, $k_0 = C^{**}(Z^i, Z^j)$ respectively.

From this it will follow that for $i < j < \omega_\gamma$, $k_0 \leqq k < \omega_\gamma$, $C_{k,k}(Z^i, Z^j)$ is blue. By this and (C), clearly for every $k < \omega_\gamma$, $l_0 < k$, $k_0 < k$

$$\{Y^i(0): i < \omega_\alpha\} \cup \{Z^j(k_0): j < \rho\}$$

is a blue subset of $\omega_{\alpha+1}$ of order-type $\omega_\alpha + \rho \geqq \omega_\alpha + \xi$, hence we prove the theorem also in the last case.

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